XXI. *Researches in the Theory of Machines. By the Rev.* HENRY MOSELEY, *M.A., F.R.S., Prof. Natural Philosophy and Astronomy, King's College, London.*

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THE work of a mechanical agent may be defined as the union of a continual pressure with a continual motion. The work of overcoming a pressure of one pound through a space of one foot, is in this country taken as the unit in terms of which any other amount of work is estimated*. The work of any pressure operating through any space is evidently measured in terms of such units, by multiplying the number of pounds in the pressure by the number of feet in the space, if the direction of the pressure be continually that in which the space is described. If not, it follows, by a simple geometrical deduction, that it is measured by the product of the number of pounds in the pressure, by the number of feet in the projection of the space described†, upon the direction of the pressure; that is, by the product of the pressure by its virtual velocity. Thus then we conclude, at once, by the principle of virtual velocities, that if a machine work under a constant equilibrium of the pressures applied to it, or if it work uniformly, then is the aggregate work of those pressures which tend to accelerate its motion, equal to the aggregate work of those which tend to retard it; and, by the principle of vis viva, that if the machine do not work under an equilibrium of the forces impressed upon it, then is the aggregate work of those which tend to accelerate the motion of the machine, greater or less than the aggregate work of those which tend to retard its motion by one-half the aggregate of the vires vivæ acquired or lost by the moving parts of the system, whilst the work is being done upon it. In no respect have the labours of the illustrious President of the Academy of Sciences more contributed to the development of the theory of

* The sense in which the term *work* is here used, will be recognised to be that in which "dynamical effect," "efficiency," "work done," "labouring force," &c. have been understood by different English writers, and "moment d'activité," "quantité d'action," "puissance mécanique," "travail," by the French. Among the latter this variety of terms has at length given place to the most intelligible and the simplest of them, "travail." The English word *work* is the obvious translation of "*travail,*" and the use of it appears to be recommended by the same considerations. M. DUPIN has proposed the application of the term "dyname" to a unit of work. The author of this paper has gladly sheltered himself from the charge of adding to the vocabulary of scientific words by assuming the term itself, "*unit of work,*" to represent concisely and conveniently enough, without translation, the idea which is attached to it.

† If the direction of the pressure remain always parallel to itself, the space described may be any finite space; if it do not, the space is understood to be so small, that the direction of the pressure may be supposed to remain parallel to itself whilst that space is described.

machines, than in the application which he has so successfully made to it of this principle of vis viva*. In the elementary discussion, however, of this principle, which is given by M. PONCELET in the Introduction to his *Mécanique Industrielle,* he has revived the term vis inertiæ (vis inertiæ, vis insita (NEWTON)), and associating with it the definitive idea of a force of resistance opposed to the acceleration or the retardation of a body's motion, he has shown (Arts. 66. and 122.) the work expended in overcoming this resistance through any space, to be measured by one-half the vis viva accumulated through the space; so that throwing into the consideration of the forces under which a machine works, the vires inertiæ of its moving elements, and observing that one-half of their aggregate vis viva is equal to the aggregate work of their vires inertiæ, it follows by the principle of virtual velocities, that the difference between the aggregate work of those forces impressed upon a machine which tend to accelerate its motion, and the aggregate work of those which tend to retard the motion, is equal to the aggregate work of the vires inertiæ of the moving parts of the machine: under which form the principle of vis viva resolves itself into the principle of virtual velocities. So many difficulties, however, oppose themselves to the introduction of the term vis inertiæ, associated with the definitive idea of an opposing force, into the discussion of questions of mechanics, and especially of practical and elementary mechanics, that it has appeared to the author of this paper desirable to avoid it. It is with this view, that in the researches which form the subject of the paper now submitted to the Society, a new interpretation is given to that function of the velocity of a moving body which is known as its vis viva; one-half that function being interpreted to represent the number of units of work accumulated in the body so long as its motion is continued, and which number of units of work it is capable of reproducing upon any resistance which may be opposed to its motion, and bring it to rest. A very simple investigation will establish the truth of this interpretation of the analytical formula represented by the term *vis viva.* Let a body whose weight is W be conceived to descend freely by gravity through a height H, and to acquire a velocity V. It will have become capable, by reason of its motion, of overcoming a certain pressure through a certain space, that is, of yielding a certain amount of work, which amount of work may be conceived to be accumulated in it. The amount of the work which it has become capable of yielding, is manifestly that which would raise another body of the same weight W, to the same vertical height H†; or it is equivalent to a number of units of work represented by W H, or (since $V^2 = 2\,g\,H$) by $\frac{1}{2}\frac{W}{g}\,.\,V^2$, that is, by one-half the vis viva. Thus the work accumulated in a body moving with the velocity V, is represented by half the vis viva, when that velocity is acquired by the action of gravity. Now the work accumulated in a body moving

* See PONCELET, Mécanique Industrielle, troisième partie.

† If a mechanical contrivance could be so interposed as to receive the whole of the work of the descending weight, and communicate it to an equal ascending weight, this last would manifestly be projected upwards with the same velocity with which the first reached the ground, and would therefore ascend to the same height.

with this velocity V, is manifestly the same under whatever circumstances that velocity may have been acquired; the effects which a body having a given weight, and moving with a given velocity, is capable of producing (the work which it is capable of yielding) being manifestly independent of the causes from the operation of which that velocity has resulted. Since then the work which a body is capable of yielding, when its velocity has been acquired by the free action of gravity, is represented by that function of its velocity which we call one-half its vis viva, it is represented by the same function when that velocity has been acquired by the action of any other force, or under any other circumstances whatever; and if the work which it is capable of yielding upon any resistance opposed to its motion be said to be accumulated in it before it encounters that resistance, then under all circumstances is the accumulated work of a moving body represented by one-half its vis viva. Giving to the term vis viva this new interpretation, the principle of vis viva, as applied to machines, may be enunciated thus:—"The difference between the aggregate work done upon the machine during any time by those forces which tend to accelerate the motion, and the aggregate work during the same time of those which tend to retard the motion, is equal to the aggregate number of units of work accumulated in the moving parts of the machine during that time if the former aggregate exceed the latter, and lost from them during that time if the former aggregate fall short of the latter." Thus, then, if the aggregate work of the forces which tend to accelerate the motion of a machine exceeds that of the forces which tend to retard it, then is the surplus work (that done upon the driving points, above that expended upon the working points) continually accumulated in the moving elements of the machine, and their motion is thereby continually accelerated. And if the former aggregate be less than the latter, then is the deficiency supplied from the work already accumulated in the moving elements, or it is lost by them, so that their motion is in this case continually retarded.

2. The moving power divides itself whilst it operates in a machine, first, into that which overcomes the prejudicial resistances of the machine, or those which are opposed by friction and other causes, uselessly absorbing the work in its transmission. Secondly, into that which accelerates the motion of the various moving parts of the machine, and which accumulates in them so long as the work done by the moving power upon it exceeds that expended upon the various resistances opposed to the motion of the machine. Thirdly, into that which overcomes the useful resistances, or those which are opposed to the motion of the machine at the working point, or points, by the useful work which is done by it. Now the aggregate number of units of useful work yielded by any machine at its working points, is less than the number received upon the machine directly from the moving power, by the number of units expended upon the prejudicial resistances, and by the number of units accumulated in the moving parts of the machine whilst the work is being done. For if ΣU_1 represent the number of units of work received upon the machine immediately from

the operation of the moving power, Σu the whole number of such units absorbed in overcoming the prejudicial resistances opposed to the working of the machine, ΣU_2 the whole useful work of the machine (or that done in producing its useful effect), and $\frac{1}{2g} \Sigma w (v_2^2 - v_1^2)$ one half the aggregate difference of the vires vivæ of the various moving parts of the machine at the commencement and termination of the period during which the work is estimated, then, by the principle of vis viva,

$$\Sigma U_1 = \Sigma U_2 + \Sigma u + \frac{1}{2g} \Sigma w (v_2^2 - v_1^2), \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

in which v_1 and v_2 represent the velocities, at the commencement and termination of the period, during which the work is estimated, of that moving element of the machine whose weight is w. But one-half the aggregate difference of the vires vivæ of the moving elements, represents the work accumulated in them during the period in respect to which the work is estimated.

3. At every period of the motion of a machine, there obtains a relation between the motion of each one of its elements, and that of every other element, so that the velocity of every other moving element of the machine may at any time be expressed by an algebraical function of the velocity of that one element, and the space traversed by it from a given period of the motion, the constants entering into which function are determined by the forms, dimensions, and combination of the elements of the machine*. If any one such element be made to move uniformly, the other elements will either move uniformly or with a periodical motion, or some of them uniformly, and others with a periodical motion. In the first case it is evident that the motion of every element will bear a given constant ratio to that of every other. In the second case, that it will bear to it a ratio which will become the same at the expiration of each given period; it is evident moreover that this given ratio between the velocities of the moving elements, will obtain constantly or periodically under a variable as well as a constant motion of the first element of the machine. Suppose the work to be estimated during a period which is a common multiple of the periods or cycles of the different moving elements. Let V_1 represent the velocity of the moving point, or first element of the machine at the commencement of this cycle or period, which is a common multiple of all the other periods, and V_2 that at its termination, and v_1 and v_2 the velocities of any other element at the commencement and termination of the same cycle or period; then $\lambda . V_1 = v_1$, $\lambda . V_2 = v_2$, where λ represents a constant quantity given in terms of the forms, dimensions, and combination of the intervening elements of the machine. The same being true of every other element, it follows that

$$\Sigma w v_1^2 = V_1^2 . \Sigma w \lambda^2, \ \Sigma w v_2^2 = V_2^2 \Sigma w \lambda^2;$$

$$\therefore \frac{1}{2} \Sigma w (v_1^2 - v_2^2) = \frac{1}{2} (V_1^2 - V_2^2) . \Sigma w \lambda^2.$$

* Professor WILLIS has determined the form of this function in respect to each of the principal elements of complex machinery, in his work recently published, entitled 'The Principles of Mechanism.'

Substituting this value in the preceding equation (1.),

$$\Sigma U_1 = \Sigma U_2 + \Sigma u + \frac{1}{2g}(V_1^2 - V_2^2)\Sigma w \lambda^2. \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

This equation expressing a relation between the work ΣU_1 done upon the moving point of a machine, and that ΣU_2 yielded at its working points, it is proposed to call the *Modulus* of the machine.

If the velocity V_1 of the moving point be constant, or if it return to the same value at the expiration of each period, then

$$V_1 = V_2, \text{ and } \Sigma U_1 = \Sigma U_2 + \Sigma u.$$

This may be called the modulus of uniform or periodical, and the other that of variable motion. The modulus is thus in respect to any machine, the particular form applicable to that machine of the above equation, and being dependent for its amount upon the amount of work Σu expended upon the friction, and other prejudicial resistances opposed to the motion of the various elements of the machine, it measures in respect to each such machine, the loss of the work due to these causes, and therefore constitutes a true standard for comparing the expenditure of moving power necessary to the production of the same effects by different machines, and (*cæteris paribus*) a true measure of the working qualities of such machines. It has been the principal object of the researches which the author proposes to submit to the Society, in this and a subsequent paper, to develope these properties of the modulus under a general form, to determine the particular moduli of some of those elements which enter most commonly into the composition of machinery, and to deduce the moduli of various compound machines, by a general method, from the moduli of their component elements.

4. Solving equation (2.) in respect to V_2, we obtain

$$V_2^2 - V_1^2 = 2g\left\{\frac{\Sigma U_1 - \Sigma U_2 - \Sigma u}{\Sigma w \lambda^2}\right\}.$$

It is evident from this equation, that any inequality between the work ΣU_1 done upon the moving point, and that $\Sigma U_2 + \Sigma u$ yielded upon the work done, and upon the prejudicial resistances, produces a greater or less variation in the velocity of the machine, according as the quantity represented by $\Sigma w \lambda^2$ is greater or less.

It is proposed to call this quantity, which has a different value under every different mechanical combination, and which is here, it is believed, first introduced into the discussion of the theory of machines, the *coefficient of equable motion*. Being determined in respect to any machine, it measures (every other consideration being excepted) the greater or less steadiness of the motion, which is maintained by that machine under a given variation of the power which impels it.

5. *General form of the Modulus of a Machine.*

Let P_1 represent the pressure upon the moving point of a machine, and $P_2 P_3 \ldots P_n$

the pressures upon its different working points, and let that relation which obtains at any period of the motion between the moving pressure P_1 and the working pressures P_2, P_3, &c., when in the state bordering upon motion, and subject to the various prejudicial resistances under which the machine works, be represented by

$$P_1 = F(P_2, P_3, \&c.) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3.)$$

Let s_1, s_2, s_3, &c. represent the spaces described in the same exceedingly small time by the points of application of P_1, P_2, &c., if these points move in the directions in which those pressures severally act, and if not let them represent the projections of these spaces on the directions of the pressures. Then are these spaces, s_2, s_3, &c., evidently related to the space s_1 by equations of the form

$$\mu_2 s_2 = s_1, \quad \mu_3 s_3 = s_1, \quad \mu_4 s_4 = s_1, \ \&c. \ \&c.,$$

where μ_2, μ_3, μ_4, &c. are certain constant quantities determined by the forms and dimensions of the moving elements of the machine and their combination, or certain functions of these and of the space s_1 which the moving point has described from the commencement of any given period of its motion. Let now u_1 represent the work of the pressure P_1 through the space s_1, u_2 that of P_2 through s_2, &c.

$$\therefore u_1 = P_1 s_1, \quad u_2 = P_2 s_2, \quad u_3 = P_3 s_3, \ \&c.$$

$$\therefore P_1 = \frac{u_1}{s_1}, \quad P_2 = \frac{\mu_2 u_2}{s_1}, \quad P_3 = \frac{\mu_3 u_3}{s_3}, \ \&c.$$

$$\therefore \frac{u_1}{s_1} = F\left(\frac{\mu_2 u_2}{s_1}, \frac{\mu_3 u_3}{s_1}, \&c.\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.)$$

Which equation,—expressing a relation between the work u_1 at the driving point, through a small increment s_1 of the space S_1 described by that point, and the work u_2, u_3, &c. yielded during the same period at the several working points—is the modulus of the machine in respect to an exceeding small motion of its elements.

If the pressures P_1, P_2, &c. remain constant during any given period of the operation of the machine, and act continually in the same directions, it is evident that the above reasoning obtains whatever may be space s_1 through which the work u_1 is done; so that the exceeding small quantities u_1, u_2, &c. s_1 may in this case be replaced by the finite quantities U_1, U_2, &c. S_1*; S_1 representing any finite space through which the work U_1 is done at the driving point, whilst the work U_2, U_3, &c. is yielded at the working points of the machine.

If the pressures P_1, P_2, P_3, &c. be variable during any given period of the continuous operation of the machine, as it respects their several amounts, or their directions, or as to both these elements, then are they (in every case presented in the operation of machinery, simply and without the interposition of any voluntary agent) functions of the spaces S_1, S_2, S_3, &c. traversed by their points of application, and therefore of the

* If the direction of the pressure P_1 be other than that in which its point of application is made to move, S_1 must be taken to represent the projection of the space described by that point on the direction of the force.

space S_1 traversed by the point of application of the moving power; so that, representing P_1 by its value $\frac{u_1}{s_1}$, we have by equation 3,

$$\frac{u_1}{s_1} = F\,(P_2,\ P_3,\ \&c.),$$

where the second member is a function of S_1. Now if the direction in which the point of application of P_1 is made to move do not coincide with the direction in which that force acts, being inclined to it in any position at an angle θ, then, since s_1 represents in this case the projection of the increment $\Delta\, S_1$ of the space described by the point of application of P_1 on the direction of that force, we have $s_1 = \Delta\, S_1 \cos\theta$; observing, therefore, that u_1 is the increment of U_1, and representing it by $\Delta\, U_1$, we have

$$\frac{u_1}{s_1} = \frac{\Delta\, U_1}{\Delta\, S_1} \cdot \frac{1}{\cos\theta} = F\,(P_2,\ P_3\ .\ \&c.),$$

and passing to the limit

$$\frac{d\,U_1}{d\,S_1} = \cos\theta\,.\,F\,(P_2,\ P_3,\ \&c.).$$

$$\therefore\ U_1 = \int \cos\theta\,.\,F\,(P_2, P_3, \&c.)\, d\,S_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5.)$$

where θ and $F\,(P_2,\ P_3,\ \&c.)$ are functions of S_1.

The work U_1 done through a given space S_1 at the driving point under the pressures P_2, P_3, &c., at the working points of the machine, is determined by this equation in terms of S_1. Now the pressure P_2 is given in terms of the work U_2 done by it, and the distance S_2 through which it is done; and S_2 is given in terms of S_1; so that P_2 is given in terms of U_2 and S_1. In like manner P_3 is given in terms of U_3 and S_1; and so of the rest. If, therefore, we substitute for P_2, P_3, &c. in the above equation their values thus determined, we shall obtain a relation between U_1, U_2, U_3, &c. and S_1, which is the modulus required.

6. There exists in every case a relation between the quantities μ_2, μ_3, &c., which will be found useful in determining the moduli of a large class of machines. Let $P_1^{(0)}$ be taken to represent that value of P_1 which would be necessary to give motion to the machine if there were no prejudicial resistances opposed to the motion of its parts; and let $F^{(0)}\,(P_2,\ P_3,\ \&c.)$ represent the corresponding value of $F\,(P_1,\ P_2,\ \&c.)$,

$$\therefore\ P_1^{(0)} = F^{(0)}\,(P_2,\ P_3,\ \&c.).$$

Also by the principle of virtual velocities, since $P_1^{(0)}$, P_2, P_3, &c. are pressures in equilibrium, we have

$$P_1^{(0)}\,.\,s_1 = P_2\,.\,s_2 + P_3\,.\,s_3 + \ldots\,;$$

substituting for s_2, s_3, &c., their values $\frac{s_1}{\mu_2}$, $\frac{s_1}{\mu_3}$, &c., and dividing by s_1,

$$\frac{P_2}{\mu_2} + \frac{P_3}{\mu_3} + \&c. = F^{(0)}\,(P_2,\ P_3,\ \&c.) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6.)$$

In that large class of machines which present but one moving and one working point, the relation between P_1 and P_2 (equation 3.) will be found to present itself under the form

$$P_1 = a\,P_2 + E; \quad \ldots \ldots \quad (7.)$$

where a is a function of the prejudicial resistances assuming a finite value, which may be represented by $a^{(0)}$, when these resistances vanish; and where E is a function of P_2 and also of the prejudicial resistances, which vanishes with them. In this case, therefore,

$$P_1^{(0)} = F^{(0)}\,(P_2, P_3, \&c.) = a^{(0)}\,P_2;$$

and by equation 6,

$$\frac{P_2}{\mu_2} = a^{(0)}\,P_2 \quad \therefore \mu_2 = \frac{1}{a^{(0)}},$$

also

$$P_1 = F\,(P_2, P_3, \&c.) = a\,P_2 + E;$$

therefore, by equation 4,

$$\frac{u_1}{s_1} = a\,\frac{\mu_2 u_2}{s_1} + E;$$

substituting for μ_2 its value $\frac{1}{a^{(0)}}$,

$$\frac{u_1}{s_1} = \frac{a}{a^{(0)}} \cdot \frac{u_2}{s_1} + E, \quad \ldots \ldots \quad (8.)$$

by which equation the modulus of the machine, in respect to an exceedingly small motion of its parts, is determined in terms of the relation expressed by equation 7, between the moving and working pressures P_1 and P_2 in the state bordering upon motion. Assuming the moving pressure to be applied in the direction of the motion of the moving point, observing that s_1, u_1, u_2 are the increments of S_1, U_1, U_2, and passing to the limit, we have by equation (8.),

$$\frac{d\,U_1}{d\,S_1} = \frac{a}{a^{(0)}} \cdot \frac{d\,U_2}{d\,S_1} + E.$$

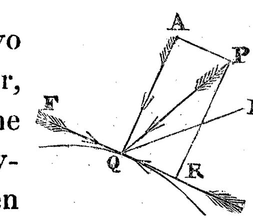
$$\therefore U_1 = \frac{a}{a^{(0)}} \,.\, U_2 + \int E\,d\,S_1, \quad \ldots \ldots \quad (9.)$$

which is the modulus of the machine. If the working pressure be *constant*, both as to its amount and its direction, E is constant, and the modulus becomes

$$U_1 = \frac{a}{a^{(0)}} \,.\, U_2 + E \,.\, S_1. \quad \ldots \ldots \quad (10.)$$

7. It remains now to consider on what general principles the relation expressed by equation 3. between the moving and the working pressures in their state bordering upon motion, may in each particular case be determined. Amongst these pressures there is, in every machine, included the resistance of one or more surfaces. Did no friction result from the pressure of the surfaces of bodies upon one another, their mutual resistance would be exerted in the direction of the common normal to their point

of contact. We know, however, by daily experience, that the resistance of no two surfaces is limited to this single direction. Friction presents itself wherever the resistance of the surfaces of solid bodies is exerted, and is, in fact, but the resolved part of that resistance in a tangent plane to the surfaces at their point of contact. And from the laws which have been proved by experiment to obtain approximately in respect to it, it follows that within the surface of a certain cone, called the cone of resistance, whose apex is at the point of contact of the surfaces, whose axis coincides with the normal, and whose angle is twice that which has for its tangent the coefficient of friction, every direction that can be taken is one in which the mutual resistances of the surfaces of contact is exerted as perfectly as in the normal direction; in fact, that any pressure (less than that which produces abrasion) being applied to the surface of an immoveable solid body by the intervention of another body moveable upon it, is sustained by the resistance of the surfaces of contact, whatever be its direction, provided only the angle which that direction makes with the perpendicular to the surfaces of contact do not exceed a certain angle, called the limiting angle of resistance of those surfaces. This is true, however great the pressure may be, within the limits of abrasion. Also, if the inclination of the pressure to the perpendicular exceed the limiting angle of resistance, then this pressure will not be sustained by the resistance of the surfaces of contact; and this is true however small the pressure may be.



Let P Q represent the direction in which the surfaces of two bodies are pressed together at Q; and let Q A be a perpendicular, or *normal* to the surfaces of contact at that point; then will the pressure P Q be sustained by the resistance of the surfaces, however *great* it may be, provided its direction lie within a certain given angle, A Q B, called the limiting angle of resistance; and it will not be sustained however small it may be, provided its direction lie without that angle. For let this pressure be represented in magnitude by P Q, and let it be resolved into two others, A Q and R Q, of which A Q is that by which it presses the surfaces together perpendicularly, and R Q that by which it tends to cause them to slide upon one another; if therefore the friction F produced by the first of these pressures exceed the second pressure R Q, then the one body will not be made to slip upon the other by this pressure P Q, however great it may be; but if the friction F, produced by the perpendicular pressure A Q, be less than the pressure R Q, then the one body will be made to slip upon the other, however small P Q may be. Let the pressure in the direction of P Q be represented by P, and the angle A Q P by θ, the perpendicular pressure in A Q is then represented by $P \cos \theta$, and therefore the friction of the surfaces of contact by $f P \cos \theta$, f representing the coefficient of friction. Moreover, the resolved pressure in the direction R Q is represented by $P \sin \theta$. The pressure P will, therefore, be sustained by the friction of the surfaces of contact, or not, according as

$$P \sin \theta \text{ is less or greater than } f P \cos \theta;$$

or dividing both sides of this inequality by P cos θ, according as

$$\tan \theta \text{ is less or greater than } f.$$

Let now the angle A Q B equal that angle whose tangent is f, and let it be represented by φ, so that $\tan \varphi = f$.

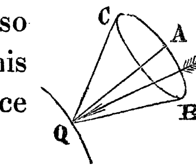
Substituting this value of f in the last inequality, it appears that the pressure P will be sustained by the friction of the surfaces of contact, or not, according as

$$\tan \theta \text{ is less or greater than } \tan \varphi,$$

that is, according as

$$\theta \text{ is less or greater than } \varphi,$$

or according as A Q P is less or greater than A Q B.



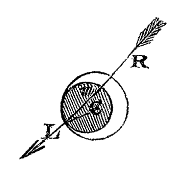
If the angle A Q B be conceived to revolve about the axis A Q, so that B Q may generate the surface of a cone B Q C, then does this cone evidently possess the properties assigned to the cone of resistance in the commencement of this section.

If the direction of the pressure P coincide with the surface of the cone, it will be sustained by the friction of the surfaces of contact, but the body to which it is applied will be upon the point of slipping on the other. The state of the equilibrium is then said to be that *bordering upon motion.*

If the pressure P admit of being applied only in a given plane, there are but two such states corresponding to those directions of P which coincide with the two intersections of the plane with the surface of the cone; these are the superior and the inferior states bordering upon motion.

Thus, then, it follows, conversely, that "when any pressure applied to a body moveable upon another which is fixed, is sustained by the resistance of the surfaces of contact of the bodies, and is in either state of the equilibrium bordering upon motion, then is the direction of that pressure, and therefore of the opposite resistance of the surface inclined to the normal at a given angle, that called the limiting angle of resistance*."

8. If any number of pressures P_1, P_2, P_3, &c. applied in the same plane to a body moveable about a cylindrical axis, be in the state bordering upon motion, then is the direction of the resistance of the axis inclined to its radius, at the point where it intersects its circumference, at an angle equal to the limiting angle of resistance. For let R represent the resultant of P_1, P_2, &c.; then, since these forces are supposed to be upon the point of causing the axis of the body to turn upon its bearings, their resultant would, if made to replace them, be also upon the point of causing the axis to turn on its bearings. Hence it follows that the direction of this resultant R cannot be through the centre C of the axis; for if it were, then the axis would be pressed by it in the



* The principle here stated was first published in the Cambridge Philosophical Transactions, vol. 5, by the author of this paper.

direction of a radius, that is, perpendicularly upon its bearings, and could not be made to turn upon them by that pressure, or to be upon the point of turning upon them. The direction of R must then be on one side of C, so as to press the axis upon its bearings in a direction R L, inclined to the perpendicular C L (at the point L where it intersects the circumference of the axis,) at a certain angle, R L C. Moreover, it is evident (by the last article) that since this force R pressing the axis upon its bearings at L is upon the point of causing it to slip upon them, this inclination R L C of R to the perpendicular C L is equal to the limiting angle of resistance of the axis and its bearings*. Now the resistance of the axis is evidently equal and opposite to the resultant R of all the forces P_1, P_2, &c. impressed upon the body. The resistance acts, therefore, in the direction L R, and is inclined to C L at an angle equal to the limiting angle of resistance.

If the radius C L of the axis be represented by ϱ, and the limiting angle of resistance C L R by φ, then is the perpendicular C m upon the resistance R from the centre C of the axis represented by $\varrho \sin \varphi$, so that the moment of R about that point is represented by $R \varrho \sin \varphi$.

9. The conditions of the equilibrium of any number of pressures in the same plane, applied to a body moveable about a cylindrical axis in the state bordering upon motion.

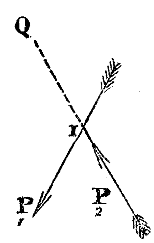
Let P_1, P_2, P_3, &c. represent these pressures, and R their resultant. Also let a_1, a_2, a_3 represent the perpendiculars let fall upon them severally from the centre of the axis, those perpendiculars being taken with the positive signs whose corresponding pressures tend to turn the system in the same direction as the pressure P_1, and those negatively which tend to turn it in the opposite direction. Also let λ represent the perpendicular distance of the direction of the resultant R from the centre of the axis, then, since R is equal and opposite to the resistance of the axis, and that this resistance and the pressures P_1, P_2, P_3, &c. are pressures in equilibrium, we have by the principle of the equality of moments,

$$P_1 a_1 + P_2 a_2 + P_3 a_3 + \&c. = \lambda R.$$

Representing, therefore, the inclinations of the directions of the pressures P_1, P_2, P_3, &c. to one another by $\iota_{1.2}$, $\iota_{1.3}$, $\iota_{2.3}$†, &c. &c., and substituting for the value of R‡,

* The side of C on which R L falls, is manifestly determined by the direction towards which the motion is about to take place. In this case it is supposed about to take place towards the left. If it had been to the right, the direction of R would have been on the opposite side of C.

† The inclination $\iota_{1.2}$ of the directions of any two pressures in the above expression, is taken on the supposition that both the pressures act *from*, or both *towards* the point in which they intersect, and not one *towards* and the other *from* that point; so that in the case represented in the accompanying figure the inclination $\iota_{1.2}$ of the pressures P_1 and P_2 represented by the arrows, is not the angle P_1 I P_2, but the angle P_1 I Q, since I Q and I P_1 are directions of these pressures, both tending *from* their point of intersection; whilst the directions of P_2 I and I P_1 are one of them *towards* that point, and the other *from* it.



‡ Poisson, Mécanique, Art. 33.

$$P_1 a_1 + P_2 a_2 + P_3 a_3 + \ldots = \lambda \left\{ \begin{array}{l} P_1^2 + P_2^2 + P_3^2 + \ldots\ldots \\ \quad + 2 P_1 P_2 \cos \iota_{1.2} + 2 P_1 P_3 \cos \iota_{1.3} + \ldots\ldots \\ \quad + 2 P_2 P_3 \cos \iota_{1.3} + 2 P_2 P_4 \cos \iota_{2.4} + \ldots\ldots \\ \quad + \&c. \&c. \end{array} \right\}^{\frac{1}{2}}.$$

$$\therefore P_1 = -\frac{P_2 a_2 + P_3 a_3 + \ldots}{a_1} + \frac{\lambda}{a_1} \left\{ \begin{array}{l} P_1^2 + 2 P_1 (P_2 \cos \iota_{1.2} + P_3 \cos \iota_{1.3} + \ldots\ldots) \\ \quad + P_2^2 + P_3^2 + P_4^2 + \ldots\ldots \\ \quad + 2 P_2 P_3 + 2 P_2 P_4 + \ldots\ldots \\ \quad + \&c. \&c. \end{array} \right\}^{\frac{1}{2}}$$

If the value of P_1 involved in this equation be expanded by LAGRANGE's theorem*, in a series ascending by powers of λ, and terms involving powers above the first be omitted, we shall obtain the following value of that quantity :—

$$P_1 = -\frac{P_2 a_2 + P_3 a_3 + \ldots}{a_1} + \left(\frac{\lambda}{a_1}\right) \left\{ \begin{array}{l} \frac{1}{a_1^2} (P_2 a_2 + P_3 a_3 + P_4 a_4 + \ldots\ldots\ldots)^2 \\ \quad - \frac{2}{a_1} (P_2 a_2 + P_3 a_3 + P_4 a_4 + \ldots\ldots\ldots). \\ \qquad (P_2 \cos \iota_{1.2} + P_3 \cos \iota_{1.3} + P_4 \cos \iota_{1.4} + \ldots\ldots) \\ \quad + P_2^2 + P_3^2 + P_4^2 + \ldots\ldots\ldots \\ \quad + 2 P_2 P_3 \cos \iota_{2.3} + 2 P_2 P_4 \cos \iota_{2.4} \\ \quad + 2 P_3 P_4 \cos \iota_{3.4} + \ldots\ldots \end{array} \right\}^{\frac{1}{2}};$$

or reducing,

$$P_1 = -\frac{P_2 a_2 + P_3 a_3 + \ldots}{a_1} + \frac{\lambda}{a_1^2} \left\{ \begin{array}{l} P_2^2 (a_1^2 - 2 a_1 a_2 \cos \iota_{1.2} + a_2^2) \\ \quad + P_3^2 (a_1^2 - 2 a_1 a_3 \cos \iota_{1.3} + a_3^2) \\ \quad + \&c. \&c. \\ \quad + 2 P_2 P_3 \{a_2 a_3 - a_1 (a_1 \cos \iota_{2.3} + a_2 \cos \iota_{1.3} + a_3 \cos \iota_{1.2})\} \\ \quad + 2 P_2 P_4 \{a_2 a_4 - a_1 (a_1 \cos \iota_{2.4} + a_2 \cos \iota_{1.4} + a_4 \cos \iota_{1.2})\} \\ \quad + \&c. \&c. \end{array} \right\}^{\frac{1}{2}}.$$

Now $a_1^2 - 2 a_1 a_2 \cos \iota_{1.2} + a_2^2$ represents the square of the line joining the feet of the perpendiculars a_1 and a_2 let fall from the centre of the axis upon P_1 and P_2; similarly $a_1^2 - 2 a_1 a_3 \cos \iota_{1.3} + a_3^2$ represents the square of the line joining the feet of the perpendicular let fall upon P_1 and P_3, and so of the rest. Let these lines be represented by $L_{1.2}$, $L_{1.3}$, $L_{1.4}$, &c., and let the different values of the function

$$\{a_2 a_3 - a_1 (a_1 \cos \iota_{2.3} + a_2 \cos \iota_{1.3} + a_3 \cos \iota_{1.2})\}$$

be represented by $M_{2.3}$, $M_{2.4}$, $M_{3.4}$, &c.,

$$\therefore P_1 = -\frac{P_2 a_2 + P_3 a_3 + \ldots}{a_1} + \frac{\lambda}{a_1^2} \left\{ \begin{array}{l} P_2^2 L_{1.2}^2 + P_3^2 L_{1.3}^2 + P_4^2 L_{1.4}^2 + \ldots \\ \quad + 2 P_2 P_3 M_{2.3} + 2 P_2 P_4 M_{2.4} + \ldots \end{array} \right\}^{\frac{1}{2}} \quad . \quad . \quad (11.)$$

10. The conditions of the equilibrium of three pressures P_1, P_2, P_3 in the same

* This expansion may be effected by squaring both sides of the equation, solving the quadratic in respect to P_1, neglecting powers of λ above the first, and reducing; this method is however exceedingly laborious.

plane applied to a body moveable about a fixed axis, the direction of one of them P_3 passing through the centre of the axis, and the system being in the state bordering upon motion by the preponderance of P_1*.

Let $\iota_{1.2}\ \iota_{1.3}\ \iota_{2.3}$ be taken, as in the preceding section, to represent the inclinations of the directions of the pressures P_1, P_2, P_3 to one another, and a_1, a_2 the perpendiculars let fall from the centre of the axis upon P_1, P_2; and λ the perpendicular let fall from the same point upon the resultant R of P_1, P_2, P_3. Then since R is equal and opposite to the *resistance* of the axis (section 8.), and that P_3 acts through the centre of the axis, and P_1 and P_2 act to turn the system in opposite directions about that centre,

$$P_1\, a_1 - P_2\, a_2 = \lambda\, R.$$

Substituting for R its value†,

$$P_1\, a_1 - P_2\, a_2 = \lambda\, \{P_1^2 + P_2^2 + P_3^2 + 2\, P_1\, P_2 \cos \iota_{1.2} + 2\, P_1\, P_3 \cos \iota_{1.3} + 2\, P_2\, P_3 \cos \iota_{2.3}\}^{\frac{1}{2}};$$

squaring both sides of this equation and transposing,

$$P_1^2\,(a_1^2 - \lambda^2) - 2\, P_1\, \{P_2\, a_1\, a_2 + \lambda^2\,(P_2 \cos \iota_{1.2} + P_3 \cos \iota_{1.3})\}$$
$$= - P_2^2\, a_2^2 + \lambda^2\, \{P_2^2 + P_3^2 + 2\, P_2\, P_3 \cos \iota_{2.3}\};$$

solving this quadratic in respect to P_1, and omitting terms which involve powers of λ above the first,

$$P_1\, a_1^2 = P_2\, a_1\, a_2 + \lambda\, \{P_2^2\,(a_1^2 + 2\, a_1\, a_2 \cos \iota_{1.2} + a_2^2) + P_3^2\, a_1^2$$
$$+ 2\, P_2\, P_3\, a_1\,(a_2 \cos \iota_{1.3} + a_1 \cos \iota_{2.3})\}^{\frac{1}{2}};$$

or representing the line which joins the feet of the perpendiculars a_1 and a_2 by L, and the function $a_1\,(a_2 \cos \iota_{1.3} + a_1 \cos \iota_{2.3})$ by M,

$$P_1 = P_2 \left(\frac{a_2}{a_1}\right) + \frac{\lambda}{a_1^2}\, \{P_2^2\, L^2 + P_3^2\, a_1^2 + 2\, P_2\, P_3\, M\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (12.)$$

If P_3 be so small as compared with P_2, that in the expansion of the irrational quantity, terms involving powers of $\frac{P_3}{P_2}$ above the first may be neglected, the above equation will become by reduction,

$$P_1 = \left(\frac{a_2}{a_1}\right) \left\{1 + \frac{L\,\lambda}{a_1\, a_2}\right\} P_2 + \frac{M\,\lambda}{a_1^2\, L}\, P_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13.)$$

If in the expressions represented by $L_{1.2}$ and $M_{2.3}$ (section 9.) we make $a_3 = 0$, give to a_2 the negative sign (since the forces P_1 and P_2 tend to turn the system in opposite directions about the axis), and observe that, since P_2 receives an opposite direction, $\cos \iota_{2.3}$ becomes negative‡, these expressions will become identical with those represented by L and M in the preceding equation (12.), and that equation will

* This problem is here investigated by an *independent* method as a verification of the theorem established in the preceding article, and as an application of it to a case of frequent occurrence in machinery.

† Poisson, Mécanique, Art. 33.

‡ See note, p. 295.

have become identical with equation (11.), and will have supplied a verification of that equation.

If the body to which the pressures P_1, P_2, P_3 are applied have its centre of gravity in the centre of the axis about which it revolves, as is commonly the case in machines, then may its weight be supposed to act through the centre of its axis, and to be represented by P_3 in the preceding formula, so that, by that formula there is represented the relation between any two pressures P_1 and P_2 applied to such a body moveable about a fixed axis, the friction of that axis and the weight of the body being taken into account.

11. The modulus of a simple machine to which are applied one moving and one working pressure, which is moveable about a fixed axis, and has its centre of gravity in the centre of that axis, the weight of the machine being taken into account.

Let P_1 and P_2 represent the moving and working pressures on the machine, and P_3 its weight, then is the relation between these pressures in the state bordering upon motion determined by equation (12.), in which λ represents the perpendicular upon the direction of the resistance of the axis, and is therefore equal (section 8.) to $\varrho \sin \varphi$, if ϱ represents the radius of the axis, and φ the limiting angle of resistance. By the substitution of this value of λ, equation (12.) becomes

$$P_1 = P_2 \left(\frac{a_2}{a_1}\right) + \frac{\varrho \sin \phi}{a_1^2} \left\{ P_2^2 L^2 + 2 P_2 P_3 M + P_3^2 a_1^2 \right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad (14.)$$

Now it is evident that this equation is of the form assumed in equation 7, section 6, the term involving the irrational quantity being represented by E (in equation 7.), and the coefficient of P_2, $\frac{a_2}{a_1}$, by a. The value of $\frac{a_2}{a_1}$ is evidently in this case independent of the prejudicial resistances, so that $a_{(0)} = \frac{a_2}{a_1}$, and $\frac{a}{a_{(0)}} = 1$. Assuming, therefore, the direction of the moving pressure P_1 to be the same with that in which its point of application is made to move, representing by θ the angle through which that point has at any time revolved, and observing that $\frac{d S_1}{d \theta} = a_1$, we have by equation 9,

$$U_1 = U_2 + \frac{\rho \sin \phi}{a_1} \int_0^{\theta} \{P_2^2 L^2 + 2 P_2 P_3 M + P_3 a_1^2\}^{\frac{1}{2}} d\theta, \quad . \quad . \quad . \quad . \quad (15.)$$

which is the modulus of the machine, and in which the term δ, involving the integral, represents the work lost by friction whilst the angle θ is described about the axis.

If the directions of the pressures P_1 and P_2 remain the same during the revolution of the body, and the working pressure P_2 be constant, then is the irrational quantity in the above expression constant, and the term involving the integral becomes by integration,

$$\frac{\rho \sin \phi}{a_1} \left\{ P_2^2 L^2 + 2 P_2 P_3 M + P_3^2 a_1^2 \right\}^{\frac{1}{2}} . \theta, \text{ or } \frac{\rho \sin \phi}{a_1 a_2} \left\{ P_2^2 L^2 + 2 P_2 P_3 M + P_3^2 a_1^2 \right\}^{\frac{1}{2}} . S_2,$$

(observing that $\theta a_2 = S_2$), or bringing S_2 under the radical sign,

$$\frac{\rho \sin \phi}{a_1 a_2} \left\{ P_2^2 S_2^2 L^2 + 2 P_2 S_2^2 P_3 M + P_3^2 S_2^2 a_1^2 \right\}^{\frac{1}{2}},$$

or

$$\frac{\rho \sin \phi}{a_1 a_2} \left\{ U_2^2 L^2 + 2 U_2 S_2 P_3 M + P_3^2 S_2^2 a_1^2 \right\}^{\frac{1}{2}};$$

so that in this case of a constant direction of the moving pressure, and a constant amount and direction of the working pressure, the modulus becomes,

$$U_1 = U_2 + \frac{\rho \sin \phi}{a_1 a_2} \left\{ U_2^2 L^2 + 2 U_2 S_2 P_3 M + P_3^2 S_2^2 a_1^2 \right\}^{\frac{1}{2}}; \quad . \quad . \quad . \quad (16.)$$

and the work lost by friction whilst the space S_2 is described by the working point, is represented by the term involving the irrational quantity in this equation.

12. A machine working about an axis of given dimensions under two pressures, P_1 and P_2, the direction and amount of one of which P_2 are given, it is required to determine that constant direction in which the other pressure P_1 must be applied, so that the machine may be worked with the greatest economy of power.

It has been shown in the last section that the work lost by friction is represented, in the case here supposed, by the formula

$$\frac{\rho \sin \phi}{a_1 a_2} \left\{ U_2^2 L^2 + 2 U_2 S_2 P_3 M + P_3^2 S_2^2 a_1^2 \right\}^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17.)$$

The machine is evidently worked then with the greatest economy of power to yield a given amount of work, U_2, when this function is a minimum. Substituting for L^2 its value

$$a_1^2 + 2 a_1 a_2 \cos \iota_{1.2} + a_2^2,$$

and for M its value

$$a_1 \{a_2 \cos \iota_{1.3} + a_1 \cos \iota_{2.3}\} \text{ (section 10.)},$$

it becomes

$$\frac{\rho \sin \phi}{a_1 a_2} \left\{ U_2^2 (a_1^2 + 2 a_1 a_2 \cos \iota_{1.2} + a_2^2) + 2 U_2 P_3 S_2 a_1 (a_2 \cos \iota_{1.3} + a_1 \cos \iota_{2.3}) + P_3^2 S_2^2 a_1^2 \right\}^{\frac{1}{2}} (18.)$$

Now let us suppose that the perpendicular distance a_2 from the centre of the axis at which the work is done, and the inclination $\iota_{2.3}$ of its direction to the vertical, are both given, as also the space S_2 through which it is done, so that the work is given in every respect; let also the perpendicular distance a_1 at which the power is applied, be given; it is required to determine that inclination $\iota_{1.2}$ of the power to the work which will under these circumstances give to the above function its minimum value, and which is, therefore, consistent with the most economical working of the machine.

Collecting all the terms in the function (18.) which contain (on the above suppositions) only constant quantities, and representing their sum

$$U_2^2 (a_1^2 + a_2^2) + 2 P_3 S_2 a_1^2 (U_2 \cos \iota_{2.3} + P_3 S_2)$$

by C^2, it becomes

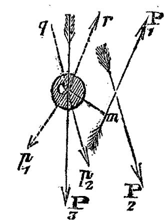
$$\frac{\rho \sin \phi}{a_1 a_2} \left\{ 2 a_1 a_2 U_2 (U_2 \cos \iota_{1.2} + P_3 S_2 \cos \iota_{1.3}) + C^2 \right\}^{\frac{1}{2}}.$$

Now C^2 being essentially positive, this quantity is a minimum when

$$2\,a_1\,a_2\,U_2\,(U_2 \cos \iota_{1.2} + P_3\,S_2 \cos \iota_{1.3})$$

is a minimum; or, observing that $U_2 = P_2\,S_2$, and dividing by the constant factor $2\,a_1\,a_2\,U_2\,S_2$; when

$$P_2 \cos \iota_{1.2} + P_3 \cos \iota_{1.3} \quad \text{is a minimum.}$$



From the centre of the axis C let lines $C\,p_1\,C\,p_2$ be drawn parallel to the directions of the pressures P_1, P_2 respectively; and whilst $C\,p_2$ and $C\,P_3$ retain their positions, let the angle $p_1\,C\,P_3$ or $\iota_{1.3}$ be conceived to increase until P_1 attains a position in which the condition $P_2 \cos \iota_{1.2} + P_3 \cos \iota_{1.3} =$ a minimum is satisfied. Now

$$p_1\,C\,P_3 = p_1\,C\,p_2 - p_2\,C\,P_3, \text{ or } \iota_{1.3} = \iota_{1.2} - \iota_{2.3};$$

substituting which value of $\iota_{1.3}$, this condition becomes

$$P_2 \cos \iota_{1.2} + P_3 \cos (\iota_{1.2} - \iota_{2.3}) = \text{a minimum},$$

or

$$P_2 \cos \iota_{1.2} + P_3 \cos \iota_{1.2} \cos \iota_{2.3} + P_3 \sin \iota_{1.2} \sin \iota_{2.3} = \text{a minimum},$$

or

$$(P_2 + P_3 \cos \iota_{2.3}) \cos \iota_{1.2} + P_3 \sin \iota_{2.3} \sin \iota_{1.2} = \text{a minimum}.$$

Let now $\dfrac{P_3 \sin \iota_{2.3}}{P_2 + P_3 \cos \iota_{2.3}} = \tan \gamma$,

so that

$$P_3 \sin \iota_{2.3} = (P_2 + P_3 \cos \iota_{2.3}) \tan \gamma$$

$$\therefore\ (P_2 + P_3 \cos \iota_{2.3}) \cos \iota_{1.2} + (P_2 + P_3 \cos \iota_{2.3}) \tan \gamma \sin \iota_{1.2} = \text{a minimum},$$

or dividing by the constant quantity $(P_2 + P_3 \cos \iota_{2.3})$, and multiplying by $\cos \gamma$,

$$\cos \iota_{1.2} \cos \gamma + \sin \iota_{1.2} \sin \gamma = \cos (\iota_{1.2} - \gamma) = \text{a minimum}.$$

$$\therefore\ \iota_{1.2} - \gamma = \pi.$$

$$\therefore\ \iota_{1.2} = \pi + \tan^{-1} \left\{ \frac{P_3 \sin \iota_{2.3}}{P_2 + P_3 \cos \iota_{2.3}} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19.)$$

To satisfy the conditions of a minimum, the angle $p_1\,C\,p_2$ must therefore be increased until it exceeds 180° by that angle γ whose tangent is represented by

$$\frac{P_3 \sin \iota_{2.3}}{P_2 + P_3 \cos \iota_{2.3}}.$$

To determine the actual direction of P_1, produce then $p_2\,C$ to q, make the angle $q\,C\,r$ equal to γ; and draw $C\,m$ perpendicular to $C\,r$, and equal to the given perpendicular distance a_1 of the direction of P_1 from the centre of the axis. If $m\,P_1$ be then drawn through the point m parallel to $C\,r$, it will be in the required direction of P_1; so that being applied in this direction, the moving pressure P_1 will work the machine with a greater economy of power than when applied in any other direction round the axis.

It is evident that since the value of the angle $\iota_{1.2}$ or $p_2\,C\,p_1$, which satisfies the condition of the greatest economy of power, or of the least resistance, is essentially greater than two right angles, P_1 and P_2 must, to satisfy that condition, both be applied on the same side of the axis. It is then a condition necessary to the most economical working of any machine (whatever may be its weight) which is moveable about a cylindrical axis under two given pressures, that the moving pressure should be applied on that side of the axis of the machine on which the resistance is overcome, or the work done. It is a further condition of the greatest economy of power in such a machine, that the direction in which the moving pressure is applied should be inclined to the vertical at an angle $\iota_{1.3}$ determined by the formula

$$\iota_{1.3} = \pi - \iota_{2.3} + \tan^{-1}\left\{\frac{P_3 \sin \iota_{2.3}}{P_2 + P_3 \cos \iota_{2.3}}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20.)$$

When $\iota_{2.3} = 0$, or when the work is done in a vertical direction, $\iota_{1.3} = \pi$, whence it follows that the moving power also must in this case be applied in a vertical direction, and on the same side of the axis as the work. When $\iota_{2.3} = \frac{\pi}{2}$, or when the work is done horizontally, $\tan \gamma = \frac{P_3}{P_2}$;

$$\therefore \ \iota_{1.2} = \pi + \tan^{-1}\left(\frac{P_3}{P_2}\right).$$

The moving power must therefore in this case be applied on the same side of the axis as the work, and at an inclination to the horizon whose tangent equals the fraction obtained by dividing the weight of the machine by the working pressure.

Since the angle $\iota_{1.2}$ is greater than π and less than $\frac{3\pi}{2}$, therefore $\cos \iota_{1.2}$ is negative; and, for a like reason, $\cos \iota_{1.3}$ is also in certain cases negative. Whence it is apparent that the function (18.) admits of a minimum value under certain conditions, not only in respect to the inclination of the moving pressure, but in respect to the distance a_1 of its direction from the centre of the axis. If we suppose the space S_1 through which the power acts whilst the given amount of work U_2 is done, to be given, and substitute in that function for the product $S_2 a_1$ its value $S_1 a_2$, and then assume the differential coefficient of the function in respect to a_1 to vanish, we shall obtain by reduction,

$$a_1 = -a_2 \cdot \frac{U_2^2 + 2 U_2 P_3 S_1 \cos \iota_{1.3} + P_3^2 S_1^2}{U_2^2 \cos \iota_{1.2} + U_2 P_3 S_1 \cos \iota_{2.3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21.)$$

If we proceed in like manner, assuming the space S_2 instead of S_1 to be constant, and substituting in the function (18.) for $S_1 a_2$ its value $S_2 a_1$, we shall obtain by reduction,

$$a_1 = -\frac{P_2 a_2}{P_2 \cos \iota_{1.2} + P_3 \cos \iota_{2.3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

It is easily seen that, if, when the values of $\iota_{1.2}$ and $\iota_{1.3}$ determined by equations 19 and 20. are substituted in these equations, the resulting values of a_1 are positive, they correspond, in the two cases, to minimum values of the function (18.), and determine completely the conditions of the greatest economy of power in the machine, in respect to the direction of the moving pressure applied to it.

13. *The Modulus of the Pulley.*

Let P_1 and P_2 be taken to represent the *moving* and *working* (or the preponderating and yielding) tensions upon the two parts of the cord passing over a pulley; let W represent its weight, a its radius measuring to the centre of the cord, ϱ the radius of its axis, and φ the limiting angle of resistance between the axis and its bearings. Then if the cord were without rigidity, we should have by equation (13.), observing that $a_1 = a_2 = a$, and substituting W for P_3, and $\varrho \sin \varphi$ for λ,

$$P_1 = \left\{1 + \frac{L\rho}{a_2} \sin \varphi\right\} P_2 + \frac{M\rho}{La^2} . W \sin \varphi.$$

But by the experiments of COULOMB (as reduced by M. PONCELET)*, it appears that the effect of the rigidity of the cord is the same as though it increased the tension P_2 so as to become $P_2\left(1 + \frac{E}{a}\right) + \frac{D}{a}$, where E and D are certain constants given in terms of the diameter of the rope. Taking into account the effect of this rigidity, the relation between P_1 and P_2 becomes therefore

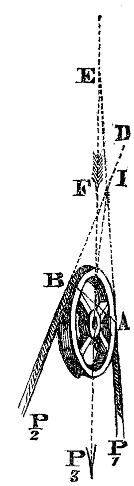
$$P_1 = \left\{1 + \frac{L\rho}{a^2} \sin \varphi\right\} \left\{P_2\left(1 + \frac{E}{a}\right) + \frac{D}{a}\right\} + \frac{M\rho}{La^2} W \sin \varphi,$$

whence by reduction we have

$$P_1 = \left(1 + \frac{E}{a}\right)\left\{1 + \frac{L\rho}{a^2} \sin \varphi\right\} P_2 + \frac{D}{a}\left\{1 + \left(\frac{L}{a^2} + \frac{MW}{LDa}\right) \varrho \sin \varphi\right\}, \quad . \quad (23.)$$

where L represents the chord of the arc embraced by the string, and M the quantity $a^2 (\cos \iota_{1.3} + \cos \iota_{2.3})$, $\iota_{1.3}$ and $\iota_{2.3}$ being the inclinations of the two parts of the string to the vertical (section 10.).

Let the accompanying figure be taken to represent the pulley with the cord passing over it, and E P_3 the direction of the weight of the pulley, supposed to act through the centre of its axis, then are the angles $\iota_{1.3}$ and $\iota_{2.3}$ represented by P_1 E P_3, and P_2 F P_3, or their *supplements*, according as the pressures P_1 and P_2 respectively act *downwards*, as shown in the figure, or *upwards*†; so that if *both* these pressures act upwards, then the cosines of both angles become negative, and the value of M is negative; whilst if *one* only acts upwards, then one term only of the value of M assumes a negative value. Let the inclination A I B of the two parts of the string be represented by 2ι, then $L = AB = 2a\cos\iota$. Substituting this value for L, and also its value $a^2 (\cos \iota_{1.3} + \cos \iota_{2.3})$ for M, and omitting terms which involve products of the exceedingly small quantities $\frac{D}{a}, \frac{E}{a}$ and $\frac{\rho}{a} \sin \varphi$, we have



$$P_1 = \left\{1 + \frac{E}{a} + \frac{2\rho}{a} \cos \iota \sin \varphi\right\} P_2 + \frac{D}{a} + \frac{W\rho(\cos \iota_{1.3} + \cos \iota_{2.3}) \sin \phi}{2a \cos \iota}.$$

* See PONCELET, Mécanique Industrielle, 128.

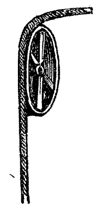
† See Note, Section 9.

Whence* we obtain for the *modulus* of the pulley,

$$U_1 = \left\{1 + \frac{E}{a} + \frac{2\rho}{a} \cos \iota \sin \varphi\right\} U_2 + \left\{\frac{D}{a} + \frac{W\rho(\cos \iota_{1.3} + \cos \iota_{2.3}) \sin \phi}{2\,a \cos \iota}\right\} S_1. \quad . \quad (24.)$$

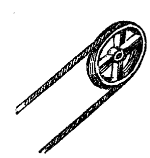
If both the strings be inclined at equal angles to the vertical, on opposite sides of it, or if $\iota_{1.3} = \iota_{2.3} = \iota$, so that $\cos \iota_{1.3} + \cos \iota_{2.3} = 2 \cos \iota$, the modulus becomes

$$U_1 = \left\{1 + \frac{E}{a} + \frac{2\rho}{a} \cos \iota \sin \varphi\right\} U_2 + \left\{\frac{D}{a} + \frac{W\rho}{a} \sin \varphi\right\} S_1. \quad . \quad . \quad . \quad . \quad (25.)$$



If one part of the cord passing over a pulley have a horizontal, and the other a vertical direction, as, for instance, when it passes into the *shaft* of a mine over the sheaf or wheel which overhangs its mouth, then one of the angles $\iota_{1.3}, \iota_{2.3}$ (equation 24.) becomes $\frac{\pi}{2}$, and the other 0 or π, according as the tension of the vertical part of the cord is upwards or downwards, so that $\cos \iota_{1.3} + \cos \iota_{2.3} = \pm 1$, the sign $\pm$ being taken according as the tension on the vertical branch of the cord is upwards or downwards: moreover in this case $\iota = \frac{\pi}{4}$, and $\cos \iota = \frac{1}{\sqrt{2}}$, therefore by equation (24.),

$$U_1 = \left\{1 + \frac{E}{a} + \frac{\rho\sqrt{2}}{a} \sin \varphi\right\} U_2 + \frac{1}{a}\left\{D \pm \frac{W\rho}{\sqrt{2}} \sin \varphi\right\} S_1. \quad . \quad . \quad . \quad . \quad (26.)$$

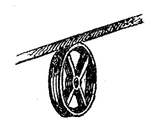


If the two parts of the cord passing over the pulley be parallel, and their common inclination to the vertical be represented by ι, so that $\iota_{1.3} = \iota_{2.3} = \iota$; then, since in this case $L = 2\,a$, we have by equation (23.), neglecting terms of more than one dimension in $\frac{E}{a}$ and $\frac{\rho}{a}$,

$$U_1 = \left\{1 + \frac{E}{a} + \frac{2\rho}{a} \sin \varphi\right\} U_2 + \frac{D}{a}\left\{1 + \left(\frac{2}{a} + \frac{W \cos \iota}{D}\right) \varrho \sin \varphi\right\}, \quad . \quad . \quad (27.)$$

in which equation, ι is to be taken greater or less than $\frac{\pi}{2}$, and therefore the sign of $\cos \iota$ is to be taken positively or negatively, according as the tensions on the cords act downwards or upwards. If the tensions are vertical, $\iota = 0$ or π, according as they act upwards or downwards, so that $\cos \iota = \pm 1$. If the parallel tensions are *horizontal*, then $\iota = \frac{\pi}{2}$, and the terms involving $\cos \iota$ in the above equations *vanish.*

If both parts of the cord passing over a pulley be in the same horizontal straight line, so that the pulley sustains no pressure resulting from the tension of the cord, but only bears its *weight*, then $\iota = \frac{\pi}{2}$, and the term involving $\cos \iota$ in equation (25.) vanishes.



It is, however, to be observed, that the weight bearing upon the axis of the pulley, is the weight of the pulley increased by the weight of the cord which it is made to support; so that if the *length* of cord supported by the pulley be represented by s, and the weight of each unit of length by μ, then is

* See Section 6, Equation 10.

the *weight* sustained by the axis of each pulley represented by $W + \mu s$. Substituting this value for W and assuming $\cos \iota = 0$ in equation (25.), we have for the modulus of the pulley in this case,

$$U_1 = \left(1 + \frac{E}{a}\right) U_2 + \frac{1}{a}\left\{D + (W + \mu s) \varrho \sin \varphi\right\} S_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28.)$$

In which equation it is supposed that although the direction of the rope on either side of each pulley is so nearly horizontal that $\cos \iota$ may be considered evanescent, yet the rope *does* so far *bend* itself over each pulley, as that its surface may adapt itself to the curved surface of the pulley, and thereby produce the whole of that resistance which is due to the rigidity of the cord.

Let it now be supposed that there is a system of n equal pulleys, or sheaves of the same dimensions, placed at equal distances in the same horizontal straight line, and sustaining each the same length s of rope.

Let U_1 represent the work done upon the cord, through the space S_1, by the moving power, or before it has passed over the first pulley of the series; U_1 the work done upon it after it has passed over the first pulley; U_2 after it has passed over the second, &c.; and U_n after it has passed over the nth pulley or sheaf; then

$$U_1 = \left(1 + \frac{E}{a}\right) U_2 + \frac{1}{a}\left\{D + (W + \mu s) \varrho \sin \varphi\right\} S_1;$$

$$U_2 = \left(1 + \frac{E}{a}\right) U_3 + \frac{1}{a}\left\{D + (W + \mu s) \varrho \sin \varphi\right\} S_1, \text{ \&c. \&c.};$$

$$U_n = \left(1 + \frac{E}{a}\right) U_{n-1} + \frac{1}{a}\left\{D + (W + \mu s) \varrho \sin \varphi\right\} S_1.$$

Eliminating the $n - 1$ quantities $U_2\ U_3 \ldots U_{n-1}$ between these n equations, and neglecting terms involving powers of $\frac{E}{a}, \frac{D}{a}, \frac{\rho}{a} \sin \varphi$ above the first, we have

$$U_1 = \left(1 + \frac{nE}{a}\right) U_n + \frac{n}{a}\left\{D + (W + \mu s) \varrho \sin \varphi\right\} S_1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29.)$$

Let us now suppose that the rope, after passing horizontally over n equal pulleys, the radius of each of which is represented by a, and its weight by W, as in the preceding case, assumes at length a vertical direction, passing over a pulley or sheaf of different dimensions, whose radius is represented by a_1, that of its axis by ϱ_1, and its weight by W_1; as for instance, when the rope of a mine descends into the shaft after having traversed the space between it and the engine, supported upon pulleys.

Let U_2 represent the work done upon the rope through the space S_1 after it has assumed the vertical direction or passed into the shaft, and let U_n represent, as before, the work done upon it after it has passed over the n horizontal pulleys, and before it passes over that which overhangs the shaft. Then by equation (26.),

$$U_n = \left\{1 + \frac{E}{a_1} + \frac{\rho_1 \sqrt{2}}{a_1} \sin \varphi\right\} U_2 + \frac{1}{a_1}\left\{D + \frac{W_1 \rho_1}{\sqrt{2}} \sin \varphi\right\} S_1.$$

Eliminating the value of U_n between this equation and equation (29.), and neglecting dimensions above the first in $\frac{E}{a}$, &c., we have

$$U_1 = \left\{1 + E\left(\frac{1}{a_1} + \frac{n}{a}\right) + \frac{\rho_1 \sqrt{2}}{a_1} \sin \varphi\right\} U_2 + \left\{D\left(\frac{1}{a_1} + \frac{n}{a}\right) + \left\{\frac{W_1 \rho_1}{a_1 \sqrt{2}} + \frac{(n W + w) \rho}{a}\right\} \sin \varphi\right\} S_1, \quad \ldots \ldots \quad (30.)$$

where w represents the whole weight $n \mu s$ of the rope supported horizontally by the pulleys. In this, as in the preceding case, it is assumed that although the rope is so nearly in the same straight line on either side of each pulley that $\cos \iota$ may be considered evanescent, yet it does so far bend as to adapt itself to the circumference of each, and thereby produce the whole of that resistance which is due to its rigidity.

HENRY MOSELEY.

King's College,
June 9, 1841.